

Negativity as Entanglement Degree of the Jaynes-Cummings Model

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Abstract

In this paper, by using the notion of negativity, we study the degree of entanglement of a two-level atom interacting with a quantized radiation field, described by the Jaynes-Cummings model (JCM). We suppose that initially the field is in a pure state and the atom is in a general mixed state. In this case the negativity fully captures the entanglement of the JCM. We investigate the case for that the initial state of the field is a coherent state. The influences of the detuning on the degree of entanglement is also examined.

Keywords: Entanglement; Negativity; Mutual entropy; Jaynes-Cummings model

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1 Introduction

Quantum entanglement is one of the most striking features of quantum mechanics which has recently attracted much attention in view of its connection with theory of quantum information and computation. It has been recognized that entanglement provides a fundamental potential resource for communication and information processing [1, 2, 3]. Entanglement is usually arising from quantum correlations between separated subsystems which can not be created by local actions on each subsystems. A pure quantum state of two or more subsystems is said to be entangled if it is not a product of states of each components. On the other hand, a bipartite mixed state ρ is said to be entangled if it can not be expressed as a convex combination of pure product states [4], otherwise, the state is separable or classically correlated. Peres [5] has shown that a necessary condition for separability of

a bipartite system is that the matrix obtained by partially transposing the density matrix ρ is still positive. Horodecki et al. [6] have shown that this condition is sufficient for separability of composite systems only when the dimension of the composite Hilbert space is $2 \otimes 2$ or $2 \otimes 3$.

Many efforts have been devoted to quantify entanglement, particularly for mixed states of a bipartite system, and a number of measures have been proposed, such as entanglement of formation, relative entropy of entanglement and negativity. For a mixed state, the entanglement of formation (EoF) is defined as the minimum of average entropy of the state over all pure state decompositions of the state [3, 7]

$$E_f(\rho) \equiv \min \sum_i p_i E(\psi_i), \quad (1)$$

where $E(\psi_i) = -\text{Tr}(\rho_i^A \ln \rho_i^A)$ is the entanglement of the pure state $|\psi_i\rangle$, and $\rho_i^A = \text{Tr}_B(|\psi_i\rangle\langle\psi_i|)$.

A class of distance measures suitable for the entanglement measures are also introduced by Vedral et al. in [8, 9], among them is the so called relative entropy of entanglement (REE), which for a given state ρ is defined by

$$E(\rho) \equiv \min_{\sigma \in \mathcal{D}} S(\rho \parallel \sigma), \quad (2)$$

where

$$S(\rho \parallel \sigma) = \text{Tr}\{\rho \ln \frac{\rho}{\sigma}\}, \quad (3)$$

is the quantum relative entropy, and \mathcal{D} is the set of all separable states. Equation (2) tells us that the amount of entanglement in ρ is its (minimum) distance from the set of separable states. It is clear that the calculation of REE as well as EoF needs the minimization procedure which, in general, is a difficult task to handle.

The Peres-Horodecki criterion for separability [5, 6] leads to a natural computable measure of entanglement,

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called negativity [10, 11, 12]. The negativity is based on the trace norm of the partial transpose ρ^{T_1} of the bipartite mixed state ρ , and measures the degree to which ρ^{T_1} fails to be positive, i.e. the absolute value of the sum of the negative eigenvalues of ρ^{T_1}

$$\mathcal{N}(\rho) \equiv \frac{\|\rho^{T_1}\|_1 - 1}{2}, \quad (4)$$

where $\|\rho^{T_1}\|_1$ denotes the trace norm of ρ^{T_1} . Vidal and Werner [12] proved that the negativity $\mathcal{N}(\rho)$ is an entanglement monotone and therefore it is a good measure of entanglement.

A lot of works have also been devoted to the preparation and measurement of entangled states between atom and radiation field. It has been recognized that the Jaynes-Cummings model [13] (JCM), comprising a two-level atom interacting with a quantized cavity mode of radiation field, plays a central role in quantum optics. The most interesting aspects of its dynamics, which has received much attention, is the possible existing of entanglement between the atom and the field. Phoenix et al [14] have demonstrated that the entangled atom-field state can be derived and have given the form of this state at all times in terms of the eigenvalues and eigenvectors of the reduced density operators. Furuichi et al have studied the entanglement of the JCM with the atom, initially in a mixed state and the field in a squeezed [15] and a coherent [16] state. Scheel et al [17] have studied the entanglement properties of the JCM in the situation for which the atom is initially in a mixed state, whereas the field has been prepared in an arbitrary thermal state. They have assessed the generated entanglement quantitatively, by evaluating the negativity and have found that, depending on the initial joint product state, three different regimes occur. In [18] the authors have studied the entanglement of the JCM in both the equilibrium and nonequilibrium time dependent ensembles. On the basis of the negativity of the partial transpose, Khemmani et al have shown that the thermal state of a coupled atomic two-level system and a field mode is never separable [19]. The evolution of the field quantum entropy and the entanglement of the atom-field in the JCM without the rotating-wave approximation have been also investigated in [20].

Considerable interest has also been devoted to the entanglement properties of the generalized JCM to include multi-photon interaction, nonlinearity of both the field and the intensity-dependent atom-field coupling, stark effect and nonresonance coupling [21, 22, 23]. The study of

entangling two mode thermal fields through the quantum erasing process, in which an atom is coupled with two mode fields via the interaction governed by the two-mode two-photon JCM has been made in [24]. In [25] the authors have considered the time-dependent JCM consists of two-mode interacting with an effective two-level atom and studied the degree of entanglement. Entanglement between two two-level atoms interacting with a single-mode field through a two-photon process [26] and a multi-photon process [27] have been studied, and on the other hand, Wang et al have investigated the entangle dynamics and entanglement distribution in a two cavity mode coupled to a two-level atom via two-photon process [28]. A three-level atom interacting with a single cavity field with an arbitrary form of the intensity-dependent coupling has been studied in [29]. The effect of dissipation on the entanglement properties in the framework of the JCM and related models have been studied [30, 31, 32, 33]. A study has been done on the entropy correlations between an atom and a single quantized cavity mode in the framework of the JCM by considering the both pure and mixed atomic and field states [34].

There exist also a considerable interest to the problem of ordering the density operators with respect to the amount of entanglement [35]. Wei et al [36] have determined families of maximally entangled states, i.e. the states which possess the maximum amount of entanglement for a given degree of mixedness. They have considered various measures of entanglement (entanglement of formation, relative entropy and negativity) and mixedness (linear entropy and von Neumann entropy) and found that the form of the maximally entangled mixed states depends on the measures used. Miranowicz et al [37] have studied the ordering of two-qubit states with respect to the Wootters's concurrence [7] and the negativity and found that the two entanglement measures can impose different ordering on the states.

In most of the previous studies on the entanglement of the JCM, quantum mutual entropy is adapted to measure the degree of entanglement of the atom and the field (for instance see [15, 16, 21, 22, 29, 32]). The quantum mutual entropy (also called index of correlation [38, 39]) for a given density matrix ρ is defined by

$$\begin{aligned} I(\rho) &\equiv \text{Tr}\{\rho (\ln \rho - \ln (\rho^A \otimes \rho^F))\} \\ &= S(\rho^A) + S(\rho^F) - S(\rho), \end{aligned} \quad (5)$$

where ρ^A and ρ^F are atomic and field density matrices, respectively, and $S(\sigma) = -\text{Tr}\sigma \ln \sigma$ is the von Neu-

mann entropy of a given state σ . Comparing with equation (3), it follows that $I(\rho)$ represents a distance between ρ and the product of marginals $\rho^A \otimes \rho^F$, i.e. it measures all correlations including classical as well as quantum correlation and does not discriminate the purely quantum entanglement from the classical correlation [8, 9, 40, 41]. It is evident that the mutual entropy is bounded $0 \leq I(\rho) \leq 2 \min[S(\rho^A), S(\rho^F)]$, and a bipartite density matrix with excessive mutual entropy, i.e. $\min[S(\rho^A), S(\rho^F)] \leq I(\rho) \leq 2 \min[S(\rho^A), S(\rho^F)]$, is entangled. This occurs precisely for quantum entangled systems and is forbidden for classical systems, therefore, a necessary condition for separability is $0 \leq I(\rho) \leq \min[S(\rho^A), S(\rho^F)]$ [34, 40]. This means that $I(\rho)$ does not fulfill the first property that we need from an entanglement measure; that is, for any separable state σ the measure of entanglement should be zero.

In this contribution, we shall turn our attention to concentrate on the negativity as the entanglement degree of the JCM. Since, in practice, it is difficult to realize an atom in a pure state, therefore we suppose that the atom is prepared, initially, in a general mixed state but, however, the field is in a pure state. It is shown that in this case the whole density matrix has rank two and supported at most on $\mathbb{C}^2 \otimes \mathbb{C}^4$ space. Since all positive partial transpose (PPT) rank two bipartite density matrices are separable [42, 43], therefore the negativity fully captures the entanglement properties of the JCM, i.e. $\mathcal{N}(\rho) = 0$ iff ρ is separable. On the other hand since the mutual entropy can not discriminate the purely quantum entanglement from the classical correlation [40, 41], therefore there is no guarantee that the value of the mutual entropy be precisely quantum entanglement, and even, it may happens that for a separable state ρ one finds a non-vanishing mutual entropy. Indeed, when two subsystems A and F are classically maximally correlated, the classical upper bound $I(\rho) = \min[S(\rho^A), S(\rho^F)]$ is saturated [40], although the system is quantum mechanically disentangled. This implies that the negativity gives a more correct estimation of the quantum entanglement. We investigate the negativity in the case that the field is in a coherent state $|\alpha\rangle$, and compare it with the mutual entropy. It is shown that only when the atom is initially in a pure state, there exist, up to a difference between the amplitudes of oscillation, a complete agreement between the negativity and the quantum mutual entropy, which the latter is twice the reduced von Neumann entropy. The influences of the detuning on the degree of entanglement

is also examined.

The organization of the paper is as follows. We start by reviewing the JCM and its solutions in section 2. The time evolution of the system is considered in section 3. Sections 4 and 5 are devoted to the calculation of the negativity in which the numerical results and their discussions are presented. The paper is concluded in section 6 with a brief conclusion.

2 The Jaynes-Cummings model

The Hamiltonian of a two-level atom interacting with a single-mode quantized radiation field is described by the JCM [13] which, within the rotating wave approximation, is one of the few exactly solvable models in quantum mechanics. Its dynamics exhibits such challenging features as collapse and revival of Rabi oscillations, and also the entanglement between the atom and the field.

The JC Hamiltonian between a two-level atom A and a single-mode quantized radiation field F is described by

$$H = H_0 + H_1, \quad (6)$$

where H_0 and H_1 , act on the product Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^F$, are given by

$$H_0 = \frac{1}{2} \hbar \omega_A \sigma_z + \hbar \omega_F a^\dagger a, \quad (7)$$

$$H_1 = \hbar g (\sigma_+ \otimes a + \sigma_- \otimes a^\dagger), \quad (8)$$

where g is the atom-field coupling constant, $\omega_A = (\epsilon_e - \epsilon_g)/\hbar$ is the atomic transition frequency, and ω_F denotes the field frequency. The atomic “spin-flip” operators $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = (\sigma_+)^\dagger$, and the atomic inversion operator $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ act on the atom Hilbert space $\mathcal{H}^A = \mathbb{C}^2$ spanned by the ground state $|g\rangle \rightarrow (0, 1)^T$ and the excited state $|e\rangle \rightarrow (1, 0)^T$. The field annihilation and creation operators a and a^\dagger satisfy the commutation relation $[a, a^\dagger] = 1$ and act on the field Hilbert space \mathcal{H}^F spanned by the photon-number states $\{|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle\}_{n=0}^\infty$, where $|0\rangle$ is the vacuum state of the field.

It is evident that the JC Hamiltonian (6) conserves the total number of excitations $K = (a^\dagger a + \sigma_z/2)$. This provides a decomposition for the system Hilbert space as $\mathcal{H} = \sum_{n=0}^\infty \oplus \mathcal{H}_n$ such that $\mathcal{H}_0 = \{|g, 0\rangle\}$ is

a one-dimensional eigensubspace of K with the eigenvalue $-1/2$, and $\mathcal{H}_{n+1}|_{n=0}^\infty = \{|e, n\rangle, |g, n+1\rangle\}$ are the two-dimensional eigensubspaces of K with the eigenvalues $n+1/2$. Corresponding to the eigensubspace $\mathcal{H}_{n+1}|_{n=0}^\infty$, there exist the eigenvalues $E_\pm^{(n)}$ and the eigenvectors $|\Phi_\pm^{(n)}\rangle$ of H as follows

$$E_\pm^{(n)} = \hbar\omega_F (n+1/2) \pm \hbar\Omega_n, \quad (9)$$

and

$$|\Phi_+^{(n)}\rangle = \sin \vartheta_n |e, n\rangle + \cos \vartheta_n |g, n+1\rangle, \quad (10)$$

$$|\Phi_-^{(n)}\rangle = \cos \vartheta_n |e, n\rangle - \sin \vartheta_n |g, n+1\rangle, \quad (11)$$

with

$$\tan \vartheta_n = \frac{2g\sqrt{n+1}}{-\Delta + 2\Omega_n}, \quad (12)$$

where $\Omega_n = \sqrt{(\Delta/2)^2 + g^2(n+1)}$ is the Rabi frequency, and $\Delta = \omega_A - \omega_F$ is the detuning parameter. In addition, corresponding to the eigensubspace \mathcal{H}_0 , there exist a negative eigenvalue given by

$$E_0 = -\frac{1}{2}\hbar\omega_A, \quad |\Phi_0\rangle = |g, 0\rangle. \quad (13)$$

Finally, taking into account equations (9), (10), (11) and (13), we arrive at the following equation for the time evolution operator $U(t)$

$$U(t) = e^{i\omega_A t/2} |g, 0\rangle\langle g, 0| + \sum_{n=0}^{\infty} \left(e^{-iE_+^{(n)}t/\hbar} |\Phi_+^{(n)}\rangle\langle\Phi_+^{(n)}| + e^{-iE_-^{(n)}t/\hbar} |\Phi_-^{(n)}\rangle\langle\Phi_-^{(n)}| \right). \quad (14)$$

3 The time evolution of the system

In order to study the entanglement properties of the JCM, let us suppose that, initially at $t=0$, the system is found in the product state

$$\rho(0) = \rho^A(0) \otimes \rho^F(0), \quad (15)$$

such that $\rho^A(0)$, the initial state of the atom, is a general mixed state with the diagonal representation

$$\rho^A(0) = \cos^2\left(\frac{\varrho}{2}\right) |g\rangle\langle g| + \sin^2\left(\frac{\varrho}{2}\right) |e\rangle\langle e|, \quad (16)$$

and $\rho^F(0)$, the initial state of the field, is a general pure state

$$\rho^F(0) = |\eta\rangle\langle\eta|, \quad |\eta\rangle = \sum_{n=0}^{\infty} b_n |n\rangle, \quad (17)$$

where the coefficients $b_n = \langle n|\eta\rangle$ are such that the state is normalized, i.e. $\sum_{n=0}^{\infty} |b_n|^2 = 1$. At the end, we will fix the coefficients b_n for the special case that the initial state of the field is a coherent state $|\alpha\rangle$. Accordingly, following the method given in [16], the final state of the system can be obtained as

$$\begin{aligned} \rho(t) = U(t)\rho(0)U^\dagger(t) = & \cos^2\left(\frac{\varrho}{2}\right) |\Psi_g(t)\rangle\langle\Psi_g(t)| \\ & + \sin^2\left(\frac{\varrho}{2}\right) |\Psi_e(t)\rangle\langle\Psi_e(t)|, \end{aligned} \quad (18)$$

where $|\Psi_g(t)\rangle$ and $|\Psi_e(t)\rangle$ are defined by

$$|\Psi_g(t)\rangle = U(t)|g, \eta\rangle = |g\rangle \otimes |\chi_1(t)\rangle + |e\rangle \otimes |\chi_2(t)\rangle, \quad (19)$$

$$|\Psi_e(t)\rangle = U(t)|e, \eta\rangle = |g\rangle \otimes |\chi_3(t)\rangle + |e\rangle \otimes |\chi_4(t)\rangle, \quad (20)$$

where the unnormalized vectors $|\chi_\alpha(t)\rangle$, ($\alpha = 1, \dots, 4$) are given by

$$\begin{aligned} |\chi_1(t)\rangle = & \sum_{n=0}^{\infty} b_n e^{-i\omega_F(n-1/2)t} \\ & \times (\cos \Omega_{n-1}t - i \cos 2\vartheta_{n-1} \sin \Omega_{n-1}t) |n\rangle, \end{aligned} \quad (21)$$

$$|\chi_2(t)\rangle = -i \sum_{n=0}^{\infty} b_{n+1} e^{-i\omega_F(n+1/2)t} \sin 2\vartheta_n \sin \Omega_n t |n\rangle, \quad (22)$$

$$|\chi_3(t)\rangle = -i \sum_{n=0}^{\infty} b_n e^{-i\omega_F(n+1/2)t} \sin 2\vartheta_n \sin \Omega_n t |n+1\rangle, \quad (23)$$

$$\begin{aligned} |\chi_4(t)\rangle = & \sum_{n=0}^{\infty} b_n e^{-i\omega_F(n+1/2)t} \\ & \times (\cos \Omega_n t + i \cos 2\vartheta_n \sin \Omega_n t) |n\rangle. \end{aligned} \quad (24)$$

Alternatively, given an orthonormal basis $\{|e_1\rangle \equiv |e\rangle, |e_2\rangle \equiv |g\rangle\} \in \mathbb{C}^2$ for the atomic Hilbert space, we can represent the density matrix $\rho(t)$ as

$$\rho(t) = \left(\begin{array}{c|c} A(t) & C(t) \\ \hline C^\dagger(t) & B(t) \end{array} \right), \quad (25)$$

where $A(t)$, $B(t)$ and $C(t)$, operators acting on the field Hilbert space, are defined by

$$A(t) = \cos^2\left(\frac{\varrho}{2}\right) |\chi_2(t)\rangle\langle\chi_2(t)| + \sin^2\left(\frac{\varrho}{2}\right) |\chi_4(t)\rangle\langle\chi_4(t)|, \quad (26)$$

$$B(t) = \cos^2\left(\frac{\rho}{2}\right)|\chi_1(t)\rangle\langle\chi_1(t)| + \sin^2\left(\frac{\rho}{2}\right)|\chi_3(t)\rangle\langle\chi_3(t)|, \quad (27)$$

$$C(t) = \cos^2\left(\frac{\rho}{2}\right)|\chi_2(t)\rangle\langle\chi_1(t)| + \sin^2\left(\frac{\rho}{2}\right)|\chi_4(t)\rangle\langle\chi_3(t)|. \quad (28)$$

Therefore, in the basis $\{|e_1\rangle, |e_2\rangle\}$, the atomic density matrix has the following matrix elements

$$\rho_{11}^A = \sum_{n=0}^{\infty} \left[\cos^2\left(\frac{\rho}{2}\right)|b_{n+1}|^2 \sin^2 2\vartheta_n \sin^2 \Omega_n t + \sin^2\left(\frac{\rho}{2}\right)|b_n|^2 (\cos^2 \Omega_n t + \cos^2 2\vartheta_n \sin^2 \Omega_n t) \right], \quad (29)$$

$$\begin{aligned} \rho_{12}^A = & -ie^{-i\omega_F t} \sum_{n=0}^{\infty} \left[\cos^2\left(\frac{\rho}{2}\right)b_{n+1}b_n^* \sin 2\vartheta_n \sin \Omega_n t \right. \\ & \times (\cos \Omega_{n-1}t + i \cos 2\vartheta_{n-1} \sin \Omega_{n-1}t) \\ & - \sin^2\left(\frac{\rho}{2}\right)b_n b_{n-1}^* \sin 2\vartheta_{n-1} \sin \Omega_{n-1}t \\ & \left. \times (\cos \Omega_n t + i \cos 2\vartheta_n \sin \Omega_n t) \right], \quad (30) \end{aligned}$$

and $\rho_{21}^A = (\rho_{12}^A)^*$, $\rho_{22}^A = 1 - \rho_{11}^A$. The field density operator also takes the following form

$$\begin{aligned} \rho^F(t) = & A(t) + B(t) \\ = & \cos^2\left(\frac{\rho}{2}\right)(|\chi_1(t)\rangle\langle\chi_1(t)| + |\chi_2(t)\rangle\langle\chi_2(t)|) \\ & + \sin^2\left(\frac{\rho}{2}\right)(|\chi_3(t)\rangle\langle\chi_3(t)| + |\chi_4(t)\rangle\langle\chi_4(t)|), \quad (31) \end{aligned}$$

4 Negativity

In the subsequent sections our goal is to quantify the entanglement of the final state (25), using the concept of the negativity defined by [12]

$$\mathcal{N}(\rho) \equiv \frac{\|\rho^{T_1}\|_1 - 1}{2}, \quad (32)$$

where ρ^{T_1} is the matrix obtained by partially transposing the density matrix ρ with respect to the first system, and $\|\rho^{T_1}\|_1$ is the trace class norm of the operator ρ^{T_1} . The trace class norm of any trace class operator A is defined by $\|A\|_1 = \text{Tr} \sqrt{A^\dagger A}$ [44], which reduces to the sum of the absolute value of the eigenvalues of A , when A is Hermitian. Therefore

$$\|\rho^{T_1}\|_1 = \sum_i |\mu_i| = \sum_i \mu_i - 2 \sum_i \mu_i^{neg} = 1 - 2 \sum_i \mu_i^{neg}, \quad (33)$$

where μ_i and μ_i^{neg} are, respectively, the eigenvalues and the negative eigenvalues of ρ^{T_1} . In the last step, we used also the fact that $\text{Tr} \rho^{T_1} = \text{Tr} \rho = 1$.

The partial transposition of $\rho(t)$ with respect to the atom in the basis $\{|e_1\rangle, |e_2\rangle\} \in \mathbb{C}^2$ is defined by

$$\rho^{T_1}(t) = \sum_{i,j=1}^2 \langle e_i | \rho(t) | e_j \rangle |e_j\rangle\langle e_i|, \quad (34)$$

where, using the representation (25), can be written in matrix form as

$$\rho^{T_1}(t) = \begin{pmatrix} A(t) & C^\dagger(t) \\ C(t) & B(t) \end{pmatrix}. \quad (35)$$

Now in order to calculate the negativity we have to obtain the eigenvalues of $\rho^{T_1}(t)$. If it was true that the set $\{|\chi_\alpha(t)\rangle\}_{\alpha=1}^4$ is linearly independent, then one can easily construct an orthonormal basis set $\{|\phi_\alpha(t)\rangle\}_{\alpha=1}^4$ from them and calculate the eigenvalues of $\rho^{T_1}(t)$ as a matrix in the orthonormal basis $\{|e_i\rangle \otimes |\phi_\alpha\rangle\}$. But this is not the case for all times, for instance at $t = 0$ we have only one independent vector $|\chi_1(0)\rangle = |\chi_4(0)\rangle = |\eta\rangle$ with $|\chi_2(0)\rangle = |\chi_3(0)\rangle = 0$. This, as we know from the initial conditions, implies that $\rho(0)$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^1$. However, due to the time evolution of the state and generation of entanglement, it is evident from numerical calculations that for $t > 0$ the dependency between vectors is destroyed and, except for some points, we have four linearly independent vectors. It follows that at any time t we can always find an N -dimensional subspace in the field Hilbert space \mathcal{H}^F such that $1 \leq N \leq 4$ denotes the number of independent vectors in the set $\{|\chi_\alpha(t)\rangle\}_{\alpha=1}^4$. Therefore the field density operator acts instantaneously on this N -dimensional subspace and $\rho(t)$ has a support on $\mathbb{C}^2 \otimes \mathbb{C}^N$, consequently, the number of nonzero eigenvalues of $\rho^{T_1}(t)$ are at most eight. On the other hand we can calculate the eigenvalues of $\rho^{T_1}(t)$ by expanding the field operators of equation (35) in the number states $\{|n\rangle\}_{n=0}^{\infty}$ as

$$\begin{aligned} A_{mn} = & \exp(-i\omega_F(m-n)t) \left[\cos^2\left(\frac{\rho}{2}\right)b_{m+1}b_{n+1}^* \right. \\ & \times \sin 2\vartheta_m \sin 2\vartheta_n \sin \Omega_m t \sin \Omega_n t \\ & + \sin^2\left(\frac{\rho}{2}\right)b_m b_n^* (\cos \Omega_m t + i \cos 2\vartheta_m \sin \Omega_m t) \\ & \left. \times (\cos \Omega_n t - i \cos 2\vartheta_n \sin \Omega_n t) \right], \quad (36) \end{aligned}$$

$$\begin{aligned}
B_{mn} = & \exp(-i\omega_F(m-n)t) \left[\sin^2\left(\frac{\theta}{2}\right) b_{m-1} b_{n-1}^* \right. \\
& \times \sin 2\vartheta_{m-1} \sin 2\vartheta_{n-1} \sin \Omega_{m-1} t \sin \Omega_{n-1} t \\
& + \cos^2\left(\frac{\theta}{2}\right) b_m b_n^* (\cos \Omega_{m-1} t - i \cos 2\vartheta_{m-1} \sin \Omega_{m-1} t) \\
& \times (\cos \Omega_{n-1} t + i \cos 2\vartheta_{n-1} \sin \Omega_{n-1} t) \left. \right], \quad (37)
\end{aligned}$$

$$\begin{aligned}
C_{mn} = & -i \exp(-i\omega_F(m-n+1)t) \times \left[\cos^2\left(\frac{\theta}{2}\right) b_{m+1} b_n^* \right. \\
& \times \sin 2\vartheta_m \sin \Omega_m t (\cos \Omega_{n-1} t + i \cos 2\vartheta_{n-1} \sin \Omega_{n-1} t) \\
& - \sin^2\left(\frac{\theta}{2}\right) b_m b_{n-1}^* \sin 2\vartheta_{n-1} \sin \Omega_{n-1} t \\
& \times (\cos \Omega_m t + i \cos 2\vartheta_m \sin \Omega_m t) \left. \right], \quad (38)
\end{aligned}$$

and $(C^\dagger)_{mn} = C_{nm}^*$. Although this representation gives an infinite-dimensional matrix for $\rho^{T_1}(t)$ which should be solved for its eigenvalues, but numerical calculations show that for large values of m and n the above matrix elements rapidly tends to zero. For instance when the field density is $|\alpha| = \sqrt{5}$ we find that $|b_n|^2$ is less than 10^{-10} for $n > 25$. Therefore the calculation of the eigenvalues of $\rho^{T_1}(t)$ can be easily handled in the truncated finite-dimensional matrix representation of $\rho^{T_1}(t)$ for some n . In all cases, however, we have only at most eight nonzero eigenvalues for $\rho^{T_1}(t)$.

Horodecki et al have shown that if the rank of a density matrix is equal to two, then the positive partial transpose is a necessary and sufficient condition for separability [42]. As we mentioned above (see equation (18)) the final state of the JCM is a rank two density matrix supported at most on $\mathbb{C}^2 \otimes \mathbb{C}^4$. This means that the PPT condition is a necessary and sufficient condition for separability of $\rho(t)$. Consequently, the negativity of the partial transpose fully captures the entanglement of $\rho(t)$, that is, $\mathcal{N}(\rho(t)) = 0$ iff $\rho(t)$ is separable.

5 Numerical results and discussion

In what follows we are going to calculate the negativity by considering special cases for the state of the field. Although $\text{rank}(\rho^{T_1}(0)) = \text{rank}(\rho(0)) = 2$, but as mentioned in the last section, due to the time evolution of the state and generation of entanglement, the rank of

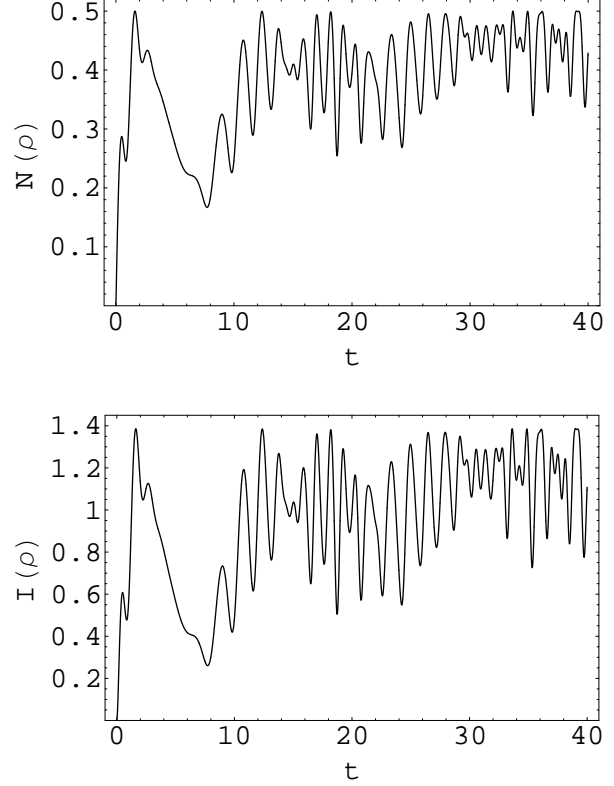


Figure 1: The time evolution of the negativity and the mutual entropy for $\alpha = \sqrt{5}$, $g = 1$, $\omega_A = 1$, $\Delta = 0$ and $\cos^2(\frac{\theta}{2}) = 0$.

$\rho^{T_1}(t)$ is no longer equal to 2 and we may have in general $2 \leq \text{rank}(\rho^{T_1}(t)) \leq 8$. Therefore it is not possible to calculate the negativity analytically and numerical calculation is required. We now suppose the initial state of the field is a coherent state $|\alpha\rangle = \sum_{n=0}^{\infty} b_n |n\rangle$ with $b_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$ [45]. This system when the atom is also in a pure state is studied by Phoenix et al [14], where they have used the von Neumann entropy of the reduced density matrix in order to describe the entanglement properties. Also by using the quantum mutual entropy, Furuichi et al [16] have studied this system with the atom in a mixed state. Our numerical results of the negativity and the mutual entropy are shown in figures (1)-(5). Figure (1) shows the negativity and the mutual entropy when the atom is initially in the excited state, i.e. $\cos^2(\frac{\theta}{2}) = 0$, and without the detuning. In this particular case the

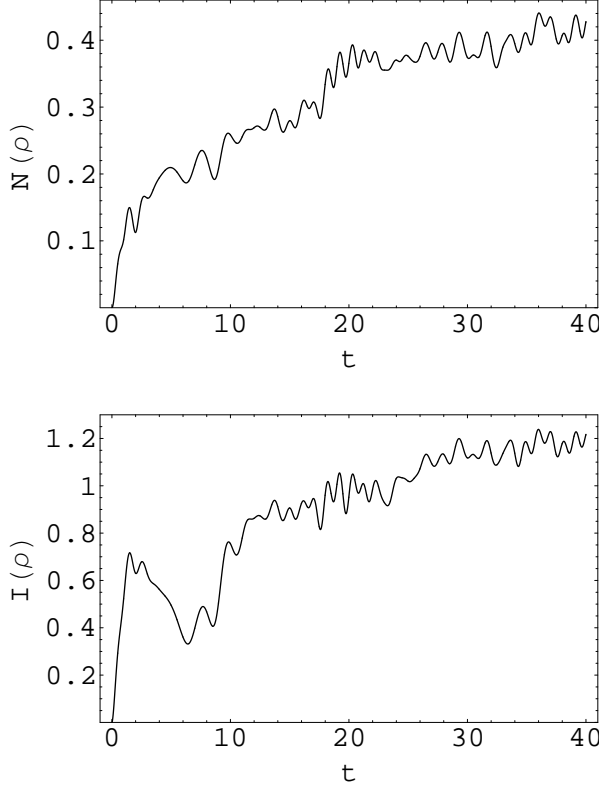


Figure 2: The time evolution of the negativity and the mutual entropy for $\alpha = \sqrt{5}$, $g = 1$, $\omega_A = 1$, $\Delta = 0$ and $\cos^2(\frac{g}{2}) = 1/2$.

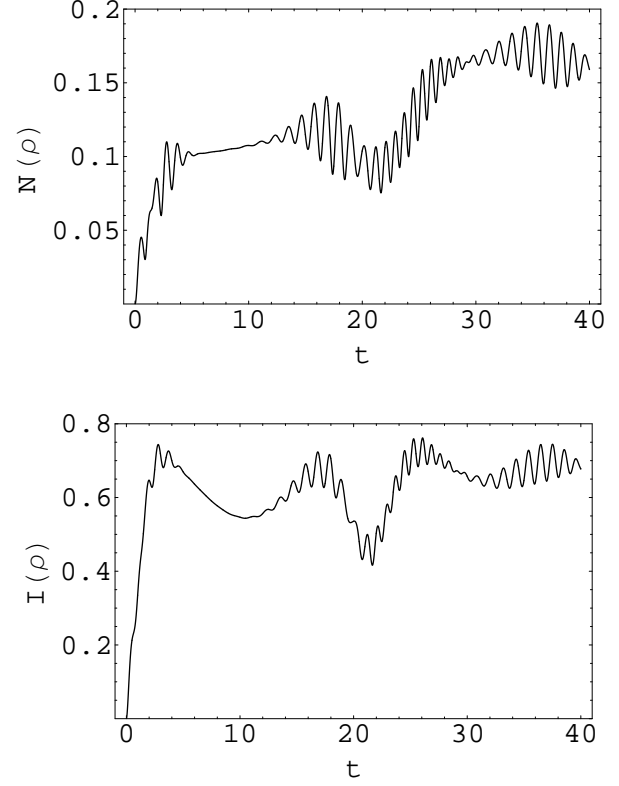


Figure 3: The time evolution of the negativity and the mutual entropy for $\alpha = \sqrt{5}$, $g = 1$, $\omega_A = 1$, $\Delta = 5$ and $\cos^2(\frac{g}{2}) = 1/2$.

state of the whole bipartite system is pure and therefore the von Neumann entropy of atom (or field) gives the entanglement of the state, and remarkably, the mutual entropy gives twice the reduced von Neumann entropy, i.e. $I(\rho(t)) = 2S(\rho(t))$. Surprisingly, up to a difference between the amplitude of oscillation, there exist a complete agreement between $\mathcal{N}(\rho)$ and $I(\rho)$ for this case. It is noteworthy that since in this case $\rho(t)$ is supported on $2 \otimes 2$ space, therefore the entanglement of formation can be calculated exactly using the Wootters's concurrence formula [7]. Let us now consider the case that $\cos^2(\frac{g}{2}) \neq 0, 1$, i.e. the initial state of the atom is not a pure state and, instead, it is a statistical mixture of the ground and the excited states. Figure (2) shows the time development of the negativity and the mutual entropy for $\cos^2(\frac{g}{2}) = 1/2$, $\Delta = 0$ and field intensity $|\alpha|^2 = 5$. Comparing $\mathcal{N}(\rho)$ and

$I(\rho)$ in this figure one finds that the agreement between two measures diminish as long as we deal with a mixed-state density matrix. It follows that the main difference between the two measures occurs during the collapse region $t \cong 3 - 7$ where the two measures do not show the same ordering for entanglement. Figures (3) and (4) are plotted the same as in figure (2) but, respectively, with the non zero detuning $\Delta = 5$ and $\Delta = 10$. Again it is apparent that the two measures do not show the same behavior.

Comparing figures (2), (3) and (4), it is clear that by increasing the detuning parameter Δ , the negativity decreases faster than the mutual entropy, therefore the relative difference between their amplitudes of oscillation increases. Numerical calculations (see figure (5)) show that for parameters of figure (4), i.e. for $\Delta = 10$, the mutual

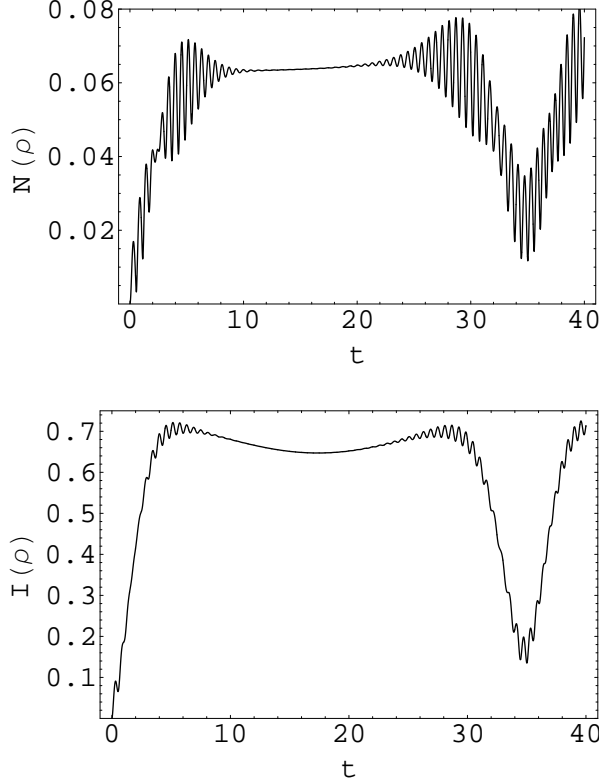


Figure 4: The time evolution of the negativity and the mutual entropy for $\alpha = \sqrt{5}$, $g = 1$, $\omega_A = 1$, $\Delta = 10$ and $\cos^2(\frac{\theta}{2}) = 1/2$.

entropy $I(\rho)$ is approximately equal to the classical upper bound $\min[S(\rho^A), S(\rho^F)]$, i.e. the bound that becomes saturated when the two subsystems A and F are classically maximally correlated. Since only the range between classical and quantum upper bounds corresponds to pure quantum entanglement [40], i.e. $\min[S(\rho^A), S(\rho^F)] \leq I(\rho) \leq 2 \min[S(\rho^A), S(\rho^F)]$, therefore it seems that for large values of the detuning, the mutual entropy evaluation of the system shows that the system is more classically correlated than quantum correlated.

6 Conclusion

In this paper we have used the negativity in order to quantify the entanglement of the JCM. We have supposed that

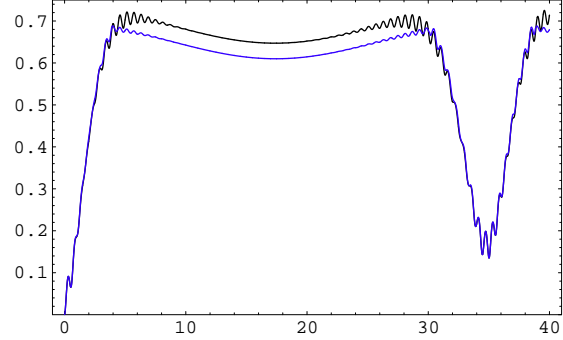


Figure 5: The time evolution of the mutual entropy (upper curve) and the classical upper bound (lower curve) for $\alpha = \sqrt{5}$, $g = 1$, $\omega_A = 1$, $\Delta = 10$ and $\cos^2(\frac{\theta}{2}) = 1/2$.

the initial state of the atom is a general mixed state, and the field is initially in a coherent state. It is shown that the state operator $\rho(t)$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^N$ where $1 \leq N \leq 4$. In this case the negativity fully captures the entanglement properties of the system. It is investigated, with the help of numerical calculations, that when the system state is pure, the negativity shows the same functionality, up to a difference in the amplitude of oscillation, with the reduced von Neumann entropy (and also mutual entropy) which is widely accepted as the entanglement measure of pure states. On the other hand for mixed states one finds that the agreement between the negativity and the mutual entropy, which have been used in the most of the previous studies on the entanglement of the JCM, is diminished. The effect of the detuning is also examined, and it is shown that the quantum entanglement evaluated by the negativity has a small value for the large detuning. We have also seen that for large values of the detuning, the mutual entropy evaluation of the system shows that the system is more classically correlated than quantum correlated. Since for any rank N bipartite system supported on $\mathbb{C}^2 \otimes \mathbb{C}^N$, the positivity of the partial transpose is necessary and sufficient condition for separability [43], therefore the negativity may be used also to study the entanglement properties of the JCM in the cases that the initial state of the field is not pure and instead it is, for instance, a non coherent mixture of the two coherent states $|\alpha\rangle$ and $|\alpha\rangle$.

At the end, it should be mentioned that various physical systems have been suggested to realize the generation of entangled states of atom and radiation, where the mi-

chromaser is one of the experimental and theoretical realization of such systems [46, 47, 48, 49]. Another fundamental system to realize the JCM and generation of entangled states is a trapped two-level ion interacting with a laser beam, where the laser beam couples the quantized internal states of the ion to the quantized center-of-mass motional states of the ion [50, 51, 52, 53, 54, 55, 56, 57].

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